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IMPROVING THE SOLUTION OF THE SYMMETRIC EIGENVALUE PROBLEM AND AN EXTENSION

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Abstract—Based on a multipoint approximation of polynomial values, we accelerate the bisection and divide-and-conquer algorithms for the symmetric eigenvalue problem. We also propose a new extension of the bisection algorithm to the unsymmetric eigenvalue problem.

In this paper we first improve the bisection algorithm for the symmetric eigenvalue problem by applying the algorithm of Pan [1], which approximates the values of a polynomial $p(x) = \sum_{i=0}^{n} p_i x^i$ on a set of m real points $x_1, \dots, x_m$, $|x_i| \leq 0$, $i = 1, \dots, m$. Then we comment on some further extensions, which may be of independent interest.

Specifically, the algorithm of Pan [1] reaches the approximations within the error bound

$$e(n) = 2^{-u} \sum_{i=0}^{n} |p_i|$$

by using $O((m + u + n) \log^2 u)$ arithmetic operations performed with infinite precision. Thus, we assume exact rational computation (with no round off errors) at this stage of the algorithm, which is performed numerically (with round off errors) at all other stages of this algorithm. Note that the order of n arithmetic operations is required to approximate or to evaluate $p(x_i)$ for each fixed single i.

The algorithm first computes the values of $p(x)$ at the Chebyshev points, $t_h = a\cos(2h + 1)/2d)\,\pi)$, $h = 0, 1, \dots, d-1$, for an appropriate $d = o(n)$, then interpolates to these d values by a polynomial $q(x)$ of degree less than d, and finally computes $q(x_i)$, which are the desired approximations to $p(x_i)$, for $i = 1, \dots, m$.

This algorithm works even if $p(x)$ is defined by a black box subroutine for its evaluation, rather than by the set of its coefficients. Due to this feature, the algorithm can be exploited to improve the bisection method for approximating the eigenvalues of a real symmetric tridiagonal matrix A. Namely, each bisection step can be reduced to the simultaneous evaluation of the sign of the characteristic polynomial $p(x) = \det(xI - A)$ at $m = O(n)$ points. We choose $u = O(\log^2 n)$ and apply the algorithm of [1] to compute the approximations $q(x_i)$ to $p(x_i)$ on these m points x_i, $i = 1, \dots, m$. For all i, for which the computed approximations $q(x_i)$ to $p(x_i)$ are such that $|q(x_i)| > e(n)$, we obtain $\operatorname{sign}(p(x_i))$. For other i, we may repeat the computation with a larger u or just evaluate $p(x_i)$.

Likewise, the divide-and-conquer (dc) algorithm for the same problem can be reduced [2,3] to

(a) multipoint evaluation of the sign of $p(x) = \det(xI - A)$ and
(b) multipoint evaluation of $p(x)$ and $p'(x)$.

Application of the algorithm of [1] at stage (a) (and possibly (b)) shall decrease the overall number of flops involved.

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REMARK 1. The above approach of this paper can be combined with the acceleration of the bisection eigenvalue algorithm proposed in [4] and based on the techniques that ensure superlinear (quadratic to cubic) convergence (right from the start) to the isolated eigenvalues or isolated clusters of k eigenvalues based on the Newton-like formula $x_{i+1} = x_i - k\,p(x_i)/p'(x_i)$, $i = 0, 1, \dots$, and on its refinements.

REMARK 2. Bini and Pan [2] use a fast (but generally numerically unstable) algorithm for simultaneous multipoint polynomial evaluation, in order to accelerate the dc algorithm; Gu and Eisenstat [5] propose to accelerate the dc algorithm by using the multipole algorithm for approximating the secular (rational) function involved.

REMARK 3. A unitary matrix U and the Hermitian matrix $U + U^H$ have the same eigenvectors; moreover, the sets of the eigenvalues of U and of $U + U^H$ are easily expressed through each other. Thus, the above approach can be extended to the eigenproblem for unitary matrices.

For the unsymmetric eigenvalue problem, the bisection method can be extended in the form of Weyl's construction, known for approximating polynomial zeros [6,7] and consisting of recursive proximity tests. Such a test computes, within the relative error $\sqrt{2} - 1$, the distance from the center of a square on the complex plane to the nearest zero of a given polynomial (in our case, to the nearest eigenvalue of a matrix A). If the test proves that a square is eigenvalue-free, the square is discarded. The test (for the eigenvalues) can be performed probabilistically, at the cost of $O(n^2 \log n)$ flops, for a Hessenberg matrix A [8]. The resulting algorithm is as robust as the bisection method and uses $O(bn^3 \log n)$ flops to approximate all (the clusters of) the eigenvalues of A within the error bound $2^{-b}\|A\|_1$. Numerical contour integration can then be applied to count the multiplicity of the eigenvalues or their numbers in each of their clusters.

REFERENCES

1. V. Pan, Linear time approximate evaluation of a polynomial at real points, Tech. Report TR 92-4, Comp. Sci. Dept. SUNYA, Albany, NY, (1992).
2. D. Bini and V. Pan, Parallel complexity of tridiagonal symmetric eigenvalue problem, In *Proc. 2nd Ann. ACM-SIAM Symp. on Discrete Algorithms*, pp. 384–393, (1991).
3. D. Bini and V. Pan, Practical improvement of the divide-and-conquer eigenvalue algorithms, *Computing* **48**, 109–123 (1992).
4. V. Pan, Accelerated solution of the symmetric eigenvalue problem, Tech. Report TR 92-6, Computer Science Dept., SUNYA, Albany, NY, (1992).
5. M. Gu and S.C. Eisenstat, private communication.
6. P. Henrici, *Applied and Computational Complex Analysis*, John Wiley, New York, (1974).
7. V. Pan, Sequential and parallel complexity of approximate evaluation of polynomial zeros, *Computers & Math. with Applications* **14** (8), 591–622 (1987).
8. J.D. Dixon, Estimating extremal eigenvalues and condition numbers of matrices, *SIAM J. of Numer. Analysis* **20** (4), 812–814 (1983).